Chapter 1 Math Foundations

P3 "The study of prob and expectation shows one way of coping with randomness and this <u>book</u> will build on probabilistic foundations to find the strongest possible links between (contingent) claims (payoff of tradable securities) and their random underlying stocks"

Purpose: to provide stochastic tools for derivative pring and hedging

Book contains very advance math and stochastic analysis tools, they are not treated rigorously

- Binomial Distribution (n, p)

n = number of trials, p = prob of successes

$n=1$ A Bernoulli RV (random variable)

$$I = \begin{cases} a & \text{with prob } p \\ b & \text{with prob } 1-p \end{cases}$$

For general integer $n > 1$

$$X = I_1 + I_2 + \cdots + I_n ,$$

where $I_1, I_2, \cdots, I_n$ are i.i.d copies of I

$X \sim \text{Binomial}(n, p)$ with possible values

$na, (n-1)a+b, \cdots, a+(n-1)b, nb$ $(n+1$ values$)$

Distribution

$$P\{X = ka + (n-k)b\} = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Its exponential version

$$Y = e^{X} = e^{I_1 + \cdots + I_n} = e^{I_1} e^{I_2} \cdots e^{I_n}$$

Y is also binomial

$$P\{ Y = [e^a]^k [e^b]^{n-k} \} = \binom{n}{k} p^k (1-p)^{n-k} .$$

– Normal (μ, σ^2)

continuous distribution with density

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} , \quad -\infty < x < \infty$$

Any linear combination of normal random variables is still a normal RV.

- Mean, Volatility and Laplace Transform

Use a continuous RV with density $f(x)$ as an example

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx \;\; \stackrel{?}{=} \sum_{i=0}^{\infty} x_i p(x_i)$$

General expectation with $h(X)$

$$E(h(X)) = \int_{-\infty}^{\infty} h(x) f(x)\,dx$$

Variance :

$$\sigma^2 = E\{(X-\mu)^2\} = E(X^2) - [E(X)]^2$$

σ = standard deviation or volatility

Laplace Transform

$h(x) = e^{-zx}$, where z is a variable

$$L_X(z) = \int_{-\infty}^{\infty} e^{-zx} f(x)\, dx = E(e^{-zX})$$

$X \sim \text{Binomial}(n, p)$ with a and b

$$E(X) = n\,E(I_1) = n\{ap + b(1-p)\}$$

$$Var(X) = Var(I_1) + \cdots + Var(I_n) = n\,Var(I_1)$$

$$= n\,(a-b)^2 p(1-p).$$

$X \sim \text{Normal}(\mu, \sigma^2)$

$E(X) = \mu$, $\quad Var(X) = \sigma^2$

The Laplace transform of $N(\mu, \sigma^2)$

$$L_X(z) = e^{-\mu z + \frac{1}{2}\sigma^2 z^2}$$

Example $S_t = S_0 e^{\mu t + \sigma W_t}$ $\underline{\mu = r - \frac{1}{2}\sigma^2}$

$$W_t \sim N(0, t)$$

Expected Return

$$E\left(\frac{S_t}{S_0}\right) = E\{e^{\mu t + \sigma W_t}\}$$

$$= e^{\mu t} E(e^{\sigma W_t}) = e^{\mu t} E(e^{-z W_t})$$

Let $z = -\sigma$

$$= e^{\mu t} L_{W_t}(z)$$

$$= e^{\mu t} e^{-0 \cdot z + \frac{1}{2} t z^2}$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t}$$

Outline of Derivation of $L_X(z)$ $\quad X \sim N(\mu, \sigma^2)$

$$L_X(z) = \int_{-\infty}^{\infty} e^{-zx} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\left[zx + \frac{(x-\mu)^2}{2\sigma^2}\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[x^2 - 2(\mu - \sigma^2 z)x + \mu^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[\left[x - (\mu - \sigma^2 z)\right]^2 + \mu^2 - (\mu - \sigma^2 z)^2\right]} dx$$

$$= e^{-\frac{1}{2\sigma^2}\left[\mu^2 - (\mu - \sigma^2 z)^2\right]} \quad !$$

— Lognormal Distribution

If $X \sim N(\mu, \sigma^2)$

$Y = e^X$ is lognormal with density

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma\, y} e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}}, \quad y > 0$$

$$E(Y) = E(e^X) = L_X(-1) = e^{\mu + \frac{1}{2}\sigma^2}$$

$$Var(Y) = E(Y^2) - [E(Y)]^2$$

$$E(Y^2) = E(e^{2X}) = L_X(-2) = e^{2\mu + 2\sigma^2}$$

Central Limit Theorem and Law of Large Numbers P4

Let $X_1, X_2, \dots, X_n, \dots$ be a i.i.d sequence

with $\mu = E(X_n)$, $\sigma^2 = Var(X_n)$

$$\overline{X}_n = \frac{X_1 + \dots + X_n}{n}, \text{ sample mean}$$

$$E(\overline{X}_n) = \mu, \quad Var(\overline{X}_n) = \frac{\sigma^2}{n}$$

CLT means the normalized sample mean

$$\frac{\overline{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \longrightarrow N(0,1), \quad \text{as } n \to \infty$$

or $\overline{X}_n \longrightarrow N\left(\mu, \frac{\sigma^2}{n}\right)$ or $X_1 + \cdots + X_n \longrightarrow N(n\mu, \sigma^2)$

$$\lim_{n \to \infty} P\{-\varepsilon < \overline{X}_n - \mu < \varepsilon\} = 1$$ for any very small ε.

Example $$I_i = \begin{cases} \frac{\mu}{n} + \frac{\sigma}{\sqrt{n}}, & p = \frac{1}{2} \\ \frac{\mu}{n} - \frac{\sigma}{\sqrt{n}} & 1-p = \frac{1}{2} \end{cases}$$

$$E(I_i) = \frac{1}{2}\left[\frac{\mu}{n} + \frac{\sigma}{\sqrt{n}} + \frac{\mu}{\sqrt{n}} - \frac{\sigma}{\sqrt{n}}\right] = \frac{\mu}{n}$$

$$\mathrm{Var}(I_i) = \left(\frac{2\sigma}{\sqrt{n}}\right)^2 \frac{1}{4} = \frac{\sigma^2}{n}$$

$X = I_1 + \cdots + I_n \qquad E(X) = \mu, \quad \mathrm{Var}(X) = \sigma^2$

$X \to N(\mu, \sigma^2)$ as $n \to \infty$ by the CLT.

Time Value of Money

Assume r is an interest rate compounded continuously

(force of interest)

1 $\qquad\qquad$ $e^{r\delta t}$

δt = length of the time period

0 $\qquad\qquad$ δt

No risk involved

Cash bond = money market account

Stock Model

Let S_T be the price of a stock at time T

$S_T = S_0 e^{X_T}$ X_T is a RV

It ensures $S_T > 0$

Two typical examples

1. $X_T \sim \text{Binomial}(n, p)$

S_T follows a Binomial model (Chapter 2)

2. $X_T \sim \text{Normal}(\mu T, \sigma^2 T)$

S_T follows a lognormal distribution

and μ is ~~the~~ the annual 'expected' return

and σ is the annual volatility.

(chapter 3)

- Actuarial Pricing vs Financial/No arbitrage Pring
- No hedging vs Hedging

Actuarial Pricing

Net Premium is the expected discounted future payoffs

Gross Premium = Net Premium + Loading

Works well for insurance policies as they can not be traded

There is no hedging

But it is not applicable to financial securities as the latter are tradable and borrowing/lending cash is available!

Example: A forward contract on S_t dilivered at T and S_t has a lognormal process.

$$S_T = S_0 e^{\mu T + \sigma\sqrt{T} Z}, \quad Z \sim N(0,1)$$

Let K be the forward price payable at time T.

Actuarial pricing implies

$$K = E_P(S_T) = S_0 e^{\mu T + \frac{1}{2}\sigma^2 T}$$

K depends on parameter μ

Since S_T is tradable, we can employ two approaches:

1. Begin with Ke^{-rT} dollars and enter a forward contract

   At time T, we have cash $K = (Ke^{-rT})e^{rT}$ dollars

   Exercise the contract and we have S_T

2. Have S_0 dollars and use the money to by one share of the stock

   At time T, we have S_T

· The no-arbitrage principle means

$$K e^{-rT} = S_0 \Rightarrow K = S_0 e^{rT}$$

$$K \neq S_0 e^{\mu T + \frac{1}{2}\sigma^2 T}$$

Chapter 2 Discrete (Binomial) Processes

Assumptions: There are two securities (tradable)

The money market account earns interest at rate r.
starting amount is B_0.
A stock is traded in discrete time with trading interval δt

Time Line

Index time	0	1	2	3	....	n	...
Actual time	0	δt	$2\delta t$	$3\delta t$	..	$n\delta t$	...

At each time i, the stock price is S_i
and there are two possibilities for stock price
movement over the next period: 'up' and 'down'.

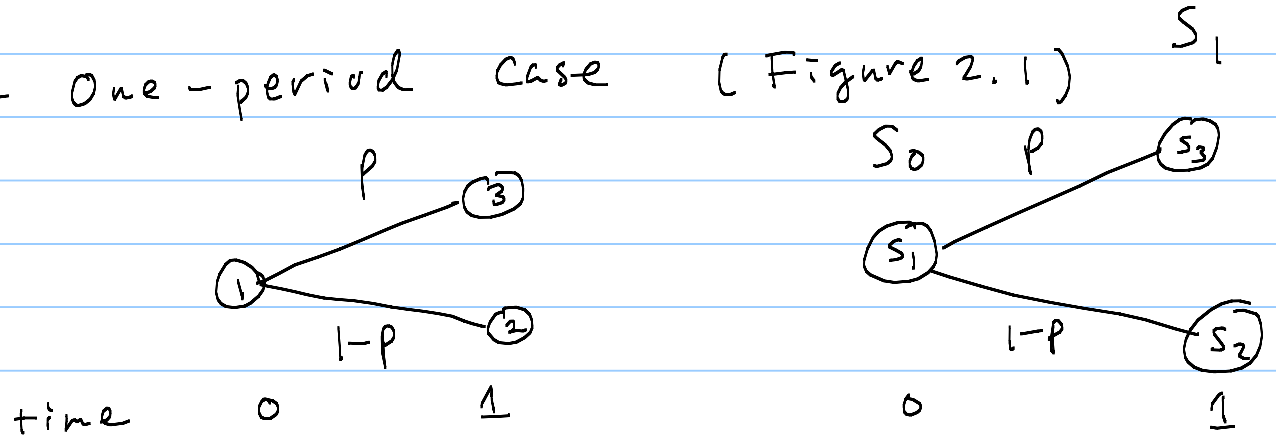
Given $S_i = s_i$

$$S_i \begin{cases} S_{2i+1} & \text{'up' with prob } p_i \\ S_{2i} & \text{'down' with prob } 1-p_i \end{cases}$$

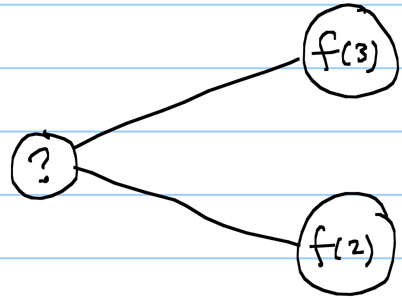
$i \qquad i+1$

p_i = the best probability estimate

— One-period case (Figure 2.1)



How to price a derivative with payoff $f(2)$ and $f(3)$



Example

$f(2) = S_2 - K$

$f(3) = S_3 - K$

use the no-arbitrage principle:

For a derivative with payoff $\underline{\Phi}(S_T)$, if we can form a self-financing portfolio such that at time T, the value of the portfolio is the same as the derivative payoff, then the price of the derivative must be equal to the initial value of the portfolio.

- Form a portfolio consisting of S_0 and B_0

Let ϕ be the number of shares of S_0 and ψ the units in the MMC

The initial value is $\phi S_0 + \psi B_0$

At time 1

$\phi S_1 + \psi B_0 e^{r\delta t}$ = payoff of the derivative.

i.e.

$$\phi S_3 + \psi B_0 e^{r \delta t} = f(3) \qquad \text{up}$$

$$\phi S_2 + \psi B_0 e^{r \delta t} = f(2) \qquad \text{down}$$

$$\phi = \frac{f(3) - f(2)}{S_3 - S_2}$$

$$\psi = e^{-r \delta t} B_0^{-1} \left[f(3) - \frac{f(3) - f(2)}{S_3 - S_2} S_3 \right]$$

The price must be

P14

$$P = \phi S_1 + \psi B_0$$

$$= \frac{f(3) - f(2)}{S_3 - S_2} S_1 + e^{-r \delta t} \left[f(3) - \frac{f(3) - f(2)}{S_3 - S_2} S_3 \right]$$

$$P = e^{-r\delta t}\left[q\, f(3) + (1-q)\, f(2)\right]$$

where $$q = \frac{e^{r\delta t} S_1 - S_2}{S_3 - S_2}$$ assuming $S_3 > S_2$

No-arbitrage condition

$$S_2 < e^{r\delta t} S_1 < S_3$$

Thus $0 < q < 1$ and

$$P = E_Q\{e^{-r\delta t} f(S_1)\}$$

P is the expected discounted payoff of the derivative, where $Q = \{q, 1-q\}$ $\Leftarrow$ Q-measure, risk-neutral measure, martingale measure

Remarks.

* The physical probability p is not in the formula
* There is a probability q, called the risk-neutral probability, such that the price of a derivative is the expected discounted payoff of the derivative.

* The present value of the stock

$e^{-r\delta t} S_1$. Its expectation under Q

$$s_1 = E_Q(e^{-r\delta t} S_1) \quad \text{or} \quad E_Q(S_1) = s_1 e^{r\delta t}$$

Proof Let $f(3) = s_3$, $f(2) = s_2$ $\square$

On the other hand

The value of the money market account B_0 at time 1

is $B_0 e^{r\delta t} = B_1 = E_Q(B_1)$

Q is risk-neutral!

Do the following

Show $S_0 = E_Q\{e^{-r\delta t} S_1\}$, $s_0 = s_1$

Show the price of a forward contract is $s_1 e^{r\delta t}$

(Exercise 2.1)

$$E_Q\{e^{-r\delta t} S_1\} = e^{-r\delta t}\left[S_3 q + S_2 (1-q) \right]$$

$$= e^{-r\delta t}\left[S_3 \frac{e^{r\delta t} S_1 - S_2}{S_3 - S_2} + S_2 \frac{S_3 - e^{r\delta t} S_1}{S_3 - S_2} \right]$$

$$= \frac{S_3 (S_1 - S_2 \cancel{e^{-r\delta t}}) + S_2 (\cancel{e^{-r\delta t}} S_3 - S_1)}{S_3 - S_2}$$

$$= \frac{S_3 S_1 - S_2 S_1}{S_3 - S_2} = S_1 \, !$$

$f(3) = S_3 - K \qquad f(2) = S_2 - K$

price at $0 = 0$

$$0 = e^{-r\delta t}\left[f(3)\, q + f(2)(1-q) \right]$$

$$f(3)\, q + f(2)(1-q) = 0$$

$$(S_3 - K)\frac{e^{r\delta t} S_1 - S_2}{\cancel{S_3 - S_2}} + (S_2 - K)\frac{S_3 - e^{r\delta t} S_1}{\cancel{S_3 - S_2}} = 0$$

$$e^{r\delta t} S_1 S_3 - \cancel{S_2 S_3} + \cancel{S_2 S_3} - e^{r\delta t} S_1 S_2$$
$$= K\left[\left(\cancel{e^{r\delta t} S_1} - S_2 \right) + \left(S_3 - \cancel{e^{r\delta t} S_1} \right) \right]$$

$$\frac{e^{r\delta t}(S_1 S_3 - S_1 S_2)}{S_3 - S_2} = K$$

$$K = S_1 e^{r\delta t}$$

Section 2.2 The Binomial Tree Model

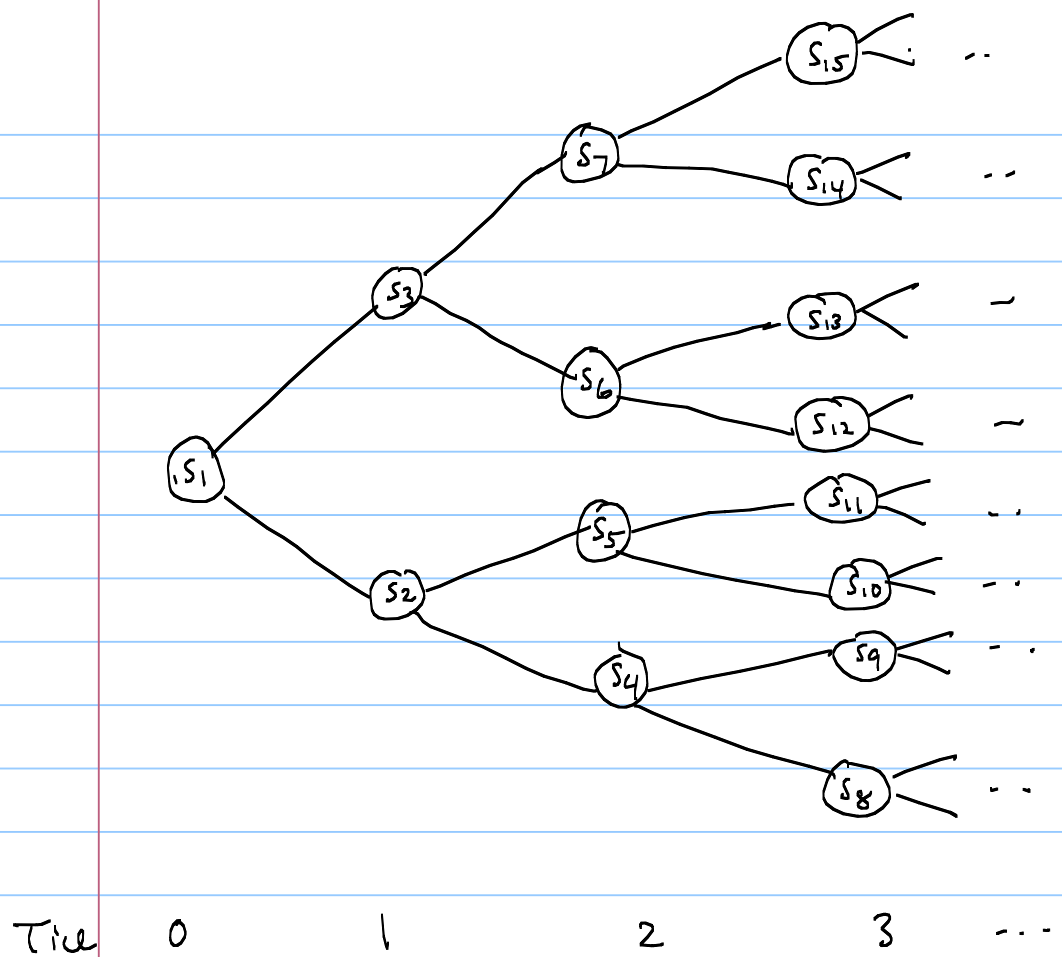
Extra assumption

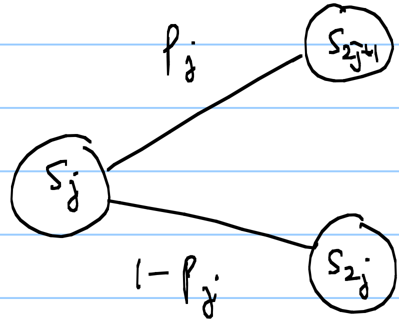
The stock price in the future will only depend on the current price but not on the prices in the past

(Markovian behavior)

Figure 2.3

S_0 S_1 S_2 S_3





p_j = physical probability

How to price a derivative maturing in the future

- Use the backwards induction (recursion)

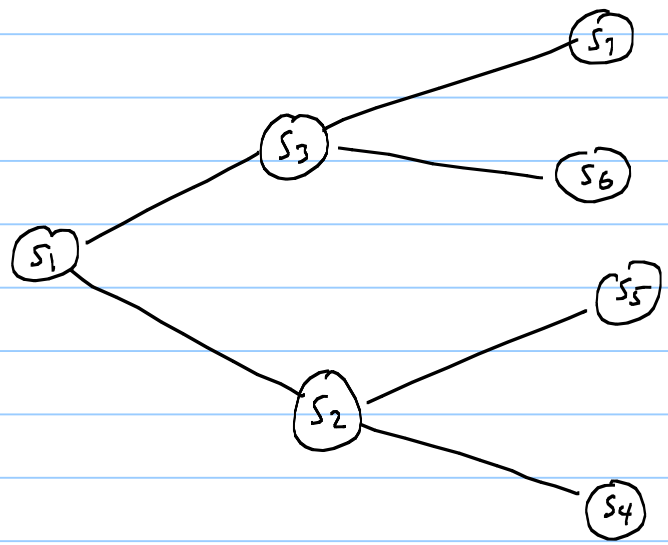
- When n is the maturing time

  Consider $(n-1, n)$ period,

  Then $(n-2, n-1)$ period, and so on.

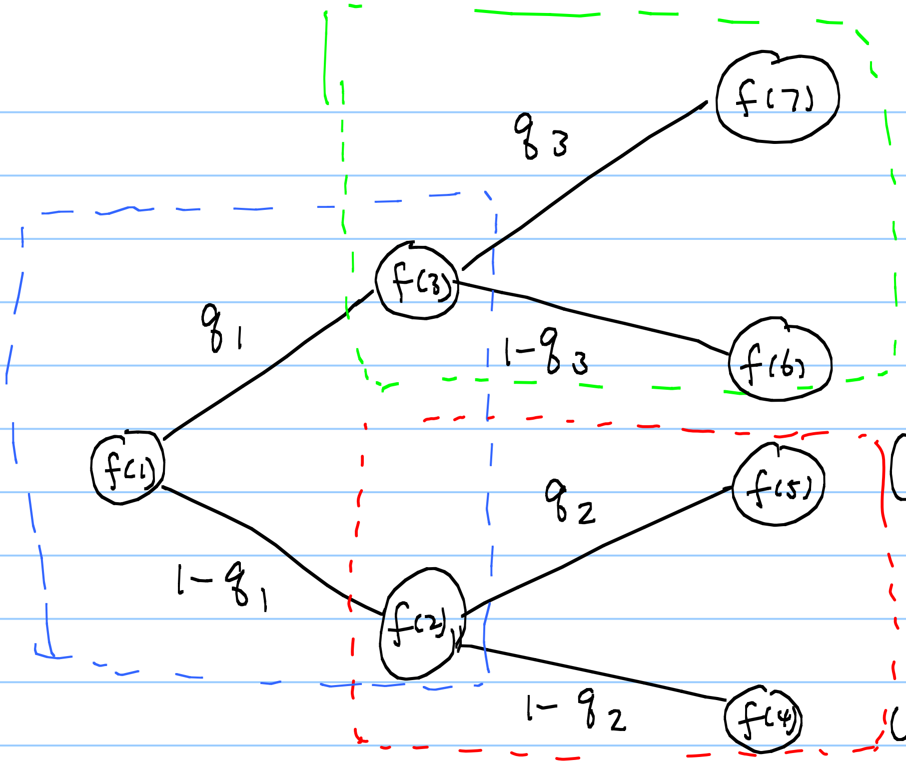
  until $(0, 1)$

An example (Figure 2.4) 3-periods $\longrightarrow$



$n=3$. and the payoff of an option at $n=3$ is

$f(4)$, $f(5)$, $f(6)$, $f(7)$



$$q_1 q_3 = Q_4$$

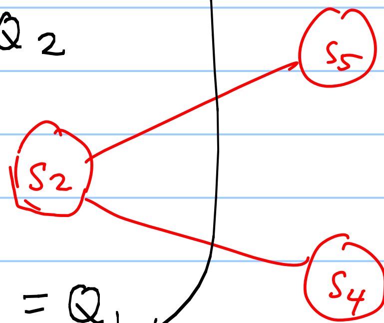
$$q_1 (1-q_3) = Q_3$$

$$(1-q_1) q_2 = Q_2$$

$$(1-q_1)(1-q_2) = Q_1$$

Risk-neutral probability measure

with



$$f(2) = e^{-r\delta t}\left[f(5)\, q_2 + f(4)(1-q_2) \right]$$

$$q_2 = \frac{e^{r\delta t} S_2 - S_4}{S_5 - S_4}$$

$$f(3) = e^{-r\delta t}\left[f(7)\, q_3 + f(6)(1-q_3)\right]$$

$$f(1) = e^{-r\delta t}\left[f(3)\, q_1 + f(2)(1-q_1)\right]$$

Alternatively, we work out the risk-neutral probability for each of all the paths

$$f(1) = e^{-r(2\delta t)}\left[f(4)\, Q_1 + f(5)\, Q_2 + f(6)\, Q_3 + f(7)\, Q_4\right]$$

$$= E_Q\left(e^{-rT} f(S_T)\right), \quad T = 2\delta t$$

$\boxed{\text{Page 21}}$

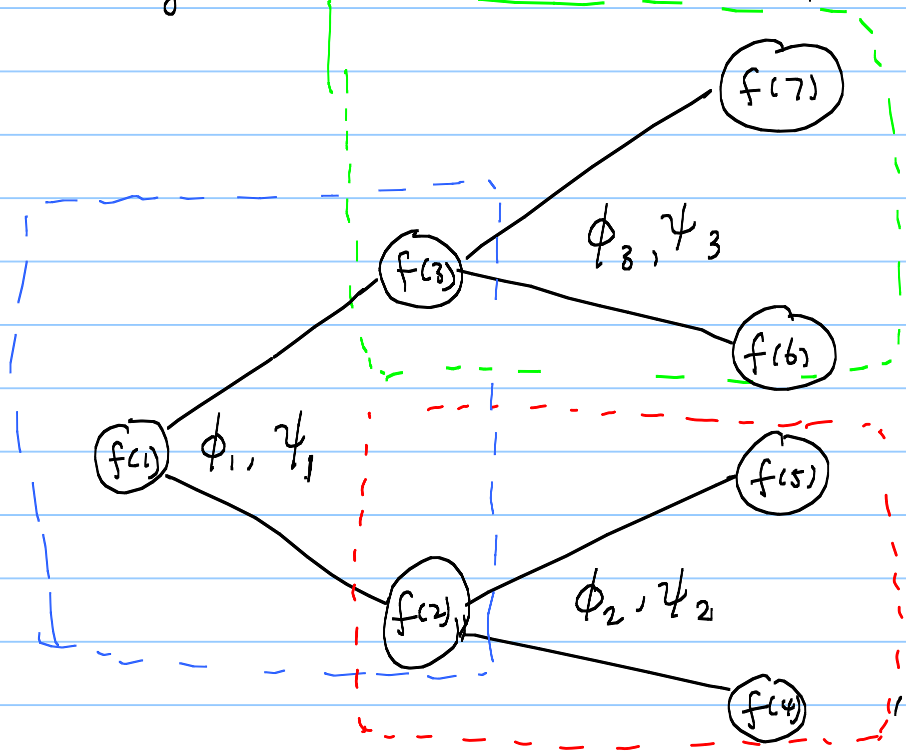
Advantage / Disadvatage

The former is tedious but it gives option prices between time 0 and time T

The latter is simpler but no price information for intermediate times

The self-financing strategy

Going back to the 1st approach



$$f(3) = \phi_3 s_3 + \psi_3 B_1 \quad , \quad B_1 = B_0 e^{r\delta t}$$

$$f(2) = \phi_2 s_2 + \psi_2 B_1$$

$$f(1) = \phi_1 s_1 + \psi_1 B_0$$ and

$$\phi_1 s_2 + \psi_1 B_1 = f(2)$$
$$= \phi_2 s_2 + \psi_2 B_1$$

$$\phi_1 s_3 + \psi_1 B_1 = f(3)$$
$$= \phi_3 s_2 + \psi_3 B_1$$

without injecting or
taking out any money!

In the end, the value of the portfolio is the
same as the payoff of the option.

* The initial value of the portfolio is
  the price of the option

* Any option can be replicated or hedged
  by a self-financing portfolio
  The hedging is perfect!

Complete Market

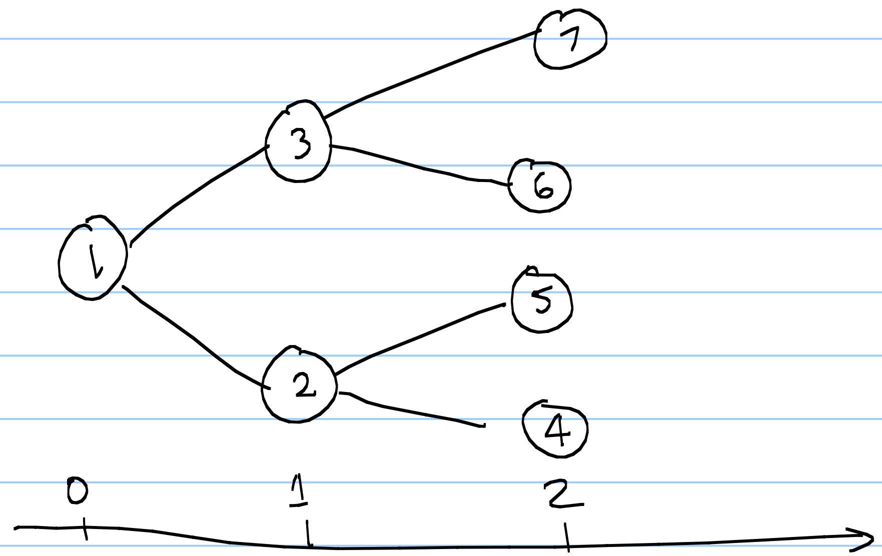
If any tradable security can be hedged perfectly using a self financing strategy, the market is complete.

If a market is complete, the price of any tradable security is uniquely determined.

And there is only one risk-neutral probability measure.

Filtration or Information Structure (P30)

Filtration = information over time



Consider S_t, $t = 0, 1, 2$.

At $t=0$, $S_0 = s_1$ No information on S_t, $t = 1, 2$.

Let $\mathcal{F}_0$ denote the info at $t=0$

$$\mathcal{F}_0 = \{1\}$$

At $t=1$, let $\mathcal{F}_1$ be the info on S_t up to time 1.

$$\mathcal{F}_1 = \{\{1,2\}, \{1,3\}\}$$

$\mathcal{F}_2$ = info up to time 2

$$\mathcal{F}_2 = \{\{1,2,4\}, \{1,2,5\}, \{1,3,6\}, \{1,3,7\}\}$$

Table 2.3

In general, if $\mathcal{F}_t$ represents info up to time t,

$\mathcal{F}_t$ is the collection of all paths up to t!

- Conditional Expectation with respect to $\mathcal{F}_t$

  Expectation calculated at time t.

  It is a random variable at t, similar to S_t

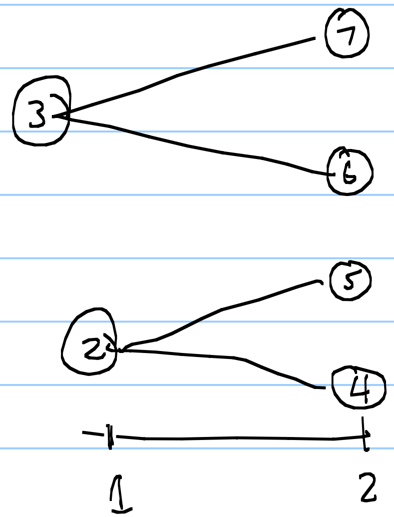
  and its value depends on the state/node at t.

Example: An option pays $f(4)$, $f(5)$, $f(6)$, or $f(7)$ at $t=2$

The price of the option at time 1

$$E_Q\left\{ e^{-r\delta t} f(S_2) \,\middle|\, \mathcal{F}_1 \right\} = \begin{cases} f(3) & \text{at } 3 \\ f(2) & \text{at } 2 \end{cases}$$

$\mathcal{F}_1 = \{ S_0, S_1 \}$



<u>Recall</u>

$$f(1) = e^{-r\delta t}\left[f(3)\, q_1 + f(2)\,(1-q_1) \right]$$

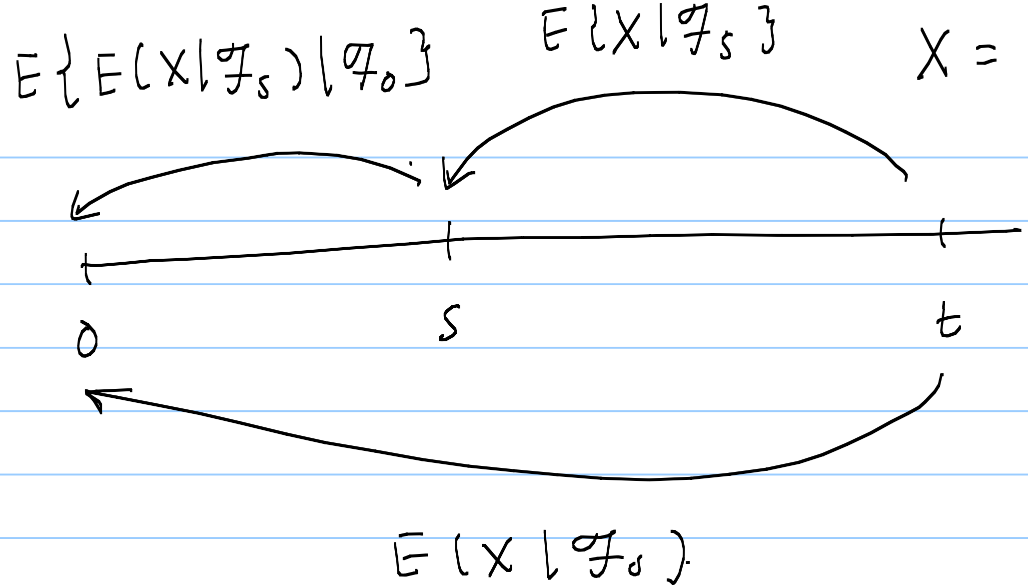
$$= E_Q\left\{ e^{-r\delta t} f(S_1) \mid \mathcal{F}_0 \right\}$$

$$f(1) = E_Q\left\{ e^{-r\delta t} E_Q\left\{ e^{-r\delta t} f(S_2) \mid \mathcal{F}_1 \right\} \mid \mathcal{F}_0 \right\}$$

$$\Rightarrow \quad \underline{E\left\{ f(S_2) \mid \mathcal{F}_0 \right\} = E\left\{ E\left\{ f(S_2) \mid \mathcal{F}_1 \right\} \mid \mathcal{F}_0 \right\}}$$

$$f(1) = E_Q\left\{ e^{-r2\delta t} f(S_2) \mid \mathcal{F}_0 \right\}$$

X = payoff of an option



$$E\{X|\mathcal{F}_0\} = E\{E\{X|\mathcal{F}_s\}|\mathcal{F}_0\}$$

Tower property (Page 34)

for $s<t$ and X is a RV on $\mathcal{F}_u$, $u>t$

$$E(X|\mathcal{F}_s) = E\{E\{X|\mathcal{F}_t\}|\mathcal{F}_s\}$$

Special cases: Recombining Binomial Trees

p23 - 28

For a general binomial tree.

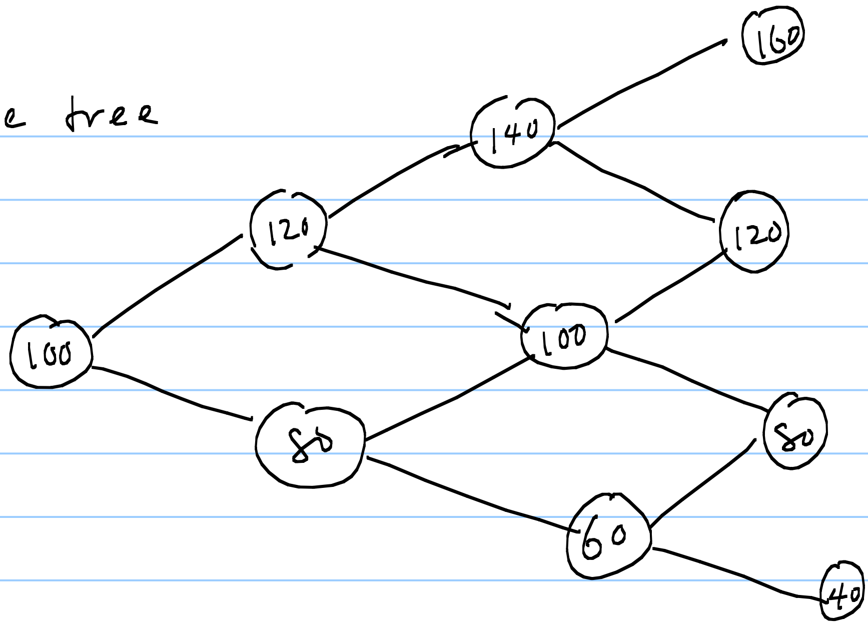
250 days = 2^{250} states too large!!

Need a feasible model.

P24 Fig 2.5

$S_0 = 100 \quad S_1 = 100 \pm 20, \quad S_2 = S_1 \pm 20, \ldots$

The tree



only depends on the number of 'up's and 'down's, but not their order

After n periods how many states $= n+1$

One 'bad' way to recombine

$$S_{i+1} = S_i \pm a$$

S_i can be negative

Also $\quad q_i = \dfrac{e^{r\delta t} S_i - (S_i - a)}{(S_i + a) - (S_i - a)}$ $\quad$ risk-neutral probability

$$= \frac{e^{r\delta t} S_i - S_i + a}{2a}$$

depends on i

NOT GOOD!

Looking for a recombing binomial tree such that $S_i > 0$ and $q_i = q$

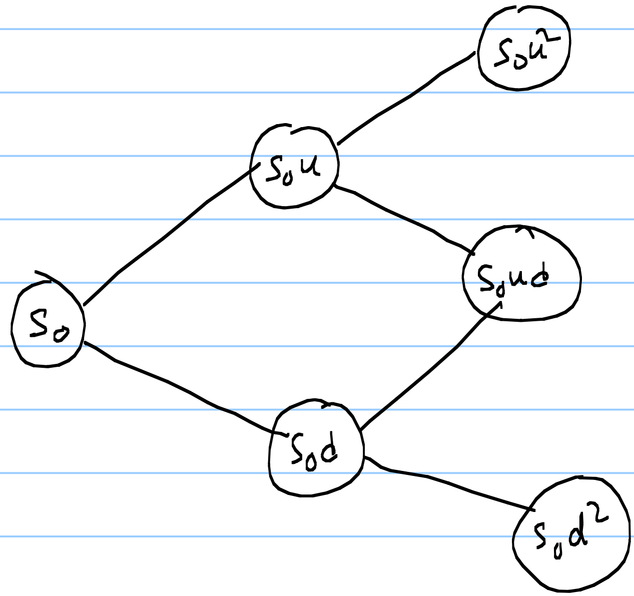
Find two positive values $u > d$ such that

$u > e^{r\delta t} > d > 0$ (no-arbitrage condition)

$$(1+x)^n =$$

and $S_{i+1} = \begin{cases} S_i u, & \text{'up'} \\ S_i d, & \text{'down'} \end{cases}$

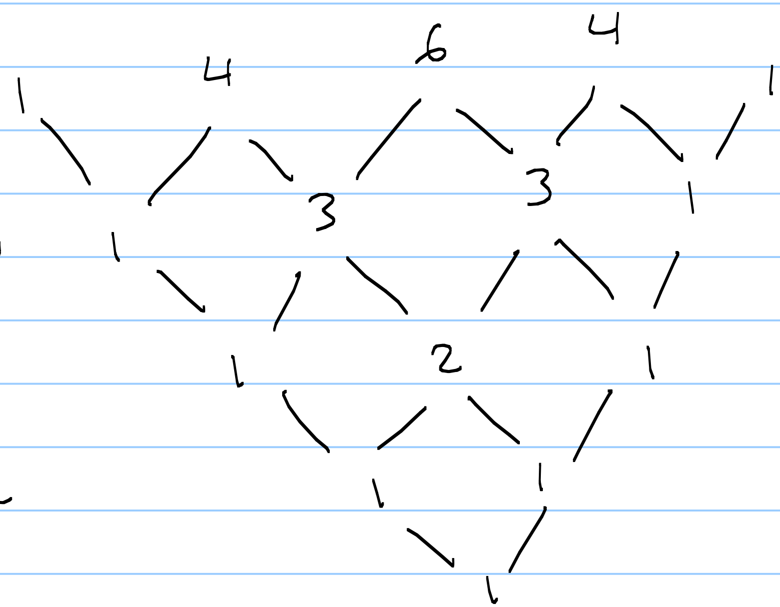
Binomial coefficients



q^2

$2q(1-q)$

$(1-q)^2$



at the end of n periods, all possible values of S_n are

$S_0 u^n$, $S_0 u^{n-1} d$, $S_0 u^{n-2} d^2$, ..., $S_0 u d^{n-1}$, $S_0 d^n$

$n+1$ values

$$q = \frac{e^{r\delta t} S_i - S_i d}{S_i u - S_i d} = \frac{e^{r\delta t} - d}{u - d}$$

Distribution of S_n under the risk-neutral prob measure

$$P\{ S_n = S_0 u^k d^{n-k} \} = \binom{n}{k} q^k (1-q)^{n-k}, \quad k = 0, 1, \ldots, n$$

The Cox-Russ-Rubinstein Formula (CRR)

for European Call option

Payoff at T $\max\{0, S_n - K\}$ $n\delta t = T$

$$\text{Price} = E_Q\{e^{-rT} \max\{0, S_n - K\}\}$$

$$= e^{-rT} \sum_{k=0}^{n} \max\{0, S_0 u^k d^{n-k} - K\} \binom{n}{k} q^k (1-q)^{n-k}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$= e^{-rT} \sum_{S_0 u^k d^{n-k} - K \geq 0} [S_0 u^k d^{n-k} - K] \binom{n}{k} q^k (1-q)^{n-k}$$

$$S_0 u^k d^{n-k} - K \geq 0 \Rightarrow k' = n-k \Rightarrow S_0 u^{n-k'} d^{k'} - K \geq 0$$

$$\Rightarrow \frac{S_0}{K} u^n \geq \left(\frac{u}{d}\right)^{k'} \Rightarrow a = \frac{\ln\left(\frac{S_0}{K}\right) + n \ln u}{\ln u - \ln d} \geq k'$$

$$= e^{-rT} \sum_{k' \leq a} S_0 u^{n-k'} d^{k'} \binom{n}{k'} q^{n-k'} (1-q)^{k'} - e^{-rT} K F(a)$$

where $F(a)$ is the d.f. of the Binomial R.V. with $p = 1-q$

The first term

$$= S_0 \sum_{k' \le a} \binom{n}{k'} e^{-rT} (uq)^{n-k'} [d(1-q)]^{k'}$$

$$= S_0 \sum_{k' \le a} \binom{n}{k'} \left[\frac{uq}{e^{r\delta t}}\right]^{n-k'} \left[\frac{d(1-q)}{e^{r\delta t}}\right]^{k'}$$

$$= S_0 \sum_{k' \le a} \binom{n}{k'} q_1^{n-k'} (1-q_1)^{k'} = S_0 F_{p_1}(a)$$

$$1 - q_1 = \frac{d(1-q)}{e^{r\delta t}} \qquad p_1 = 1 - q_1$$

$$\text{Price} = S_0 F_{p_1}(a) - e^{-rT} K F_p(a)$$

The hedging strategy : long $F_{p_1}(a)$ units of stock and borrow $e^{-rT} K F_p(a)$ dollars

Try the following $n = 3$, $u = 1.2$ $d = \frac{1}{u}$

$r = 10\%$, $\delta t = 1$ $S_0 = 100$

Price an ATM call maturing in 3 years

Section 2.4 Moving Towards Continuous-time Models

Introduce μ, annual rate of return compounded continuously or drift

σ = annualized volatility/standard deviation

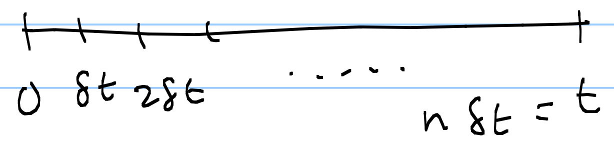
$$S_{t+\delta t} = \begin{cases} S_t\, e^{(\mu\,\delta t + \sigma\sqrt{\delta t})}, & p = \frac{1}{2} \\ S_t\, e^{(\mu\,\delta t - \sigma\sqrt{\delta t})}, & p = \frac{1}{2} \end{cases}$$

This is a recombining binomial tree with

$$u = e^{\mu \delta t + \sigma \sqrt{\delta t}}, \quad d = e^{\mu \delta t - \sigma \sqrt{\delta t}}$$

What happens to S_t when $\delta t \to 0$

Fix t first and let $n = \dfrac{t}{\delta t}$



Introduce

$$I_i = \begin{cases} \ln u, & p = \frac{1}{2} \\ \ln d, & p = \frac{1}{2} \end{cases}$$

$$= \begin{cases} \mu \delta t + \sigma \sqrt{\delta t}, & p = \frac{1}{2} \\ \mu \delta t - \sigma \sqrt{\delta t}, & p = \frac{1}{2} \end{cases}$$

$$S_{t+\delta t} = \begin{cases} S_t \, e^{\mu \delta t + \sigma \sqrt{\delta t}} \\ S_t \, e^{\mu \delta t - \sigma \sqrt{\delta t}} \end{cases} = S_t \, e^{I}$$

$$S_t = S_0 \, e^{I_1 + I_2 + \cdots + I_n}$$

$$X = I_1 + I_2 + \cdots + I_n$$

$$E(X) = n\, E(I_1) = n \, \frac{\mu \delta t + \sigma \sqrt{\delta t} + \mu \delta t - \sigma \sqrt{\delta t}}{2} = n \mu \delta t$$

$$= \mu t$$

$$Var(X) = n \, Var(I_1)$$

$$= n \left[(\mu \delta t + \sigma \sqrt{\delta t}) - (\mu \delta t - \sigma \sqrt{\delta t}) \right]^2 \frac{1}{2} \cdot \frac{1}{2}$$

$$= n\left(2\sigma\sqrt{\delta t}\right)^2 \cdot \frac{1}{4} = n\sigma^2\delta t = \sigma^2 t$$

$$\lim_{\substack{n\to\infty \\ \delta t\to 0}} X = \lim_{n\to\infty}\left(I_1 + \cdots + I_n\right) = N(\mu t, \sigma^2 t)$$

(CLT)

$$S_0 \quad \lim_{\delta t\to 0} S_t = S_0 e^{\mu t + \sigma\sqrt{t}\, Z}, \quad Z \sim N(0,1)$$

under the physical probability (P-measure)

The risk-neutral measure

$$q = \frac{e^{r\delta t} - d}{u - d}$$

$$= \frac{e^{r\delta t} - e^{\mu\delta t - \sigma\sqrt{\delta t}}}{e^{\mu\delta t + \sigma\sqrt{\delta t}} - e^{\mu\delta t - \sigma\sqrt{\delta t}}} \neq p$$

$$E(X) = n\,E(I_1) = n\left\{(\mu\delta t + \sigma\sqrt{\delta t})q + (\mu\delta t - \sigma\sqrt{\delta t})(1-q)\right\}$$

$$= n\left\{\mu\delta t - \sigma\sqrt{\delta t} + 2\sigma\sqrt{\delta t}\,q\right\}$$

$$\lim_{\delta t \to 0} E(X) = (r - \frac{1}{2}\sigma^2)t$$

$$\lim_{\delta t \to 0} Var(X) = \sigma^2 t$$

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t + \sigma\sqrt{t}\,Z}, \quad Z \sim N(0,1)$$

under the risk-neutral probability measure!